

Gapless Surfaces in Anisotropic Superfluids

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We demonstrate when p-wave pairing occurs between species whose free Fermi surfaces are mismatched the gap generally vanishes over a two-dimensional surface. We present detailed calculations of condensation energy, superfluid density (Meissner mass) and specific heat for such states. We also consider stability against separation into mixed phases. According to several independent criteria that can be checked at weak coupling, the resulting “breached” state appears to be stable over a substantial range of parameters. The simple models we consider are homogeneous in position space, and break rotation symmetry spontaneously. They should be realizable in cold atom systems.

I. INTRODUCTION

Recently there has been considerable interest in a class of possible new states featuring coexistence of superfluid and normal components. These states arise when there are interactions favoring pairing between fermions that have Fermi surfaces of different size. For s-wave superfluidity, which has mostly been considered, a breach in the pairing occurs in the pairing of fermions which have momenta whose magnitudes lie between two values p_{\pm} . Two separate two-dimensional spherical Fermi surfaces, corresponding to gapless modes, open up at $|p| = p_{\pm}$; while paired fermions with momenta outside the breach provide a coexisting superfluid condensate.

The possibility of superfluidity coexisting with gapless states at momenta that span a two-dimensional surface originally was suggested by Sarma^{1,2}. For spherically symmetric (s-wave) interactions, as he considered, a state of this type naturally suggests itself, and a pairing solution can be found^{1,2,3,4,5}. The stability of the resulting state against phase separation⁶ or the appearance of a tachyon in the gauge field (negative squared Meissner mass)^{7,8,9} is delicate, however. It appears to require some combination of unequal masses, momentum-dependent pairing interactions, and long-range neutrality constraints^{10,11}.

Cold atom systems with mismatched free Fermi surfaces naturally arise when the system contains different species of atoms, or identical atoms with different spin states in an external magnetic field. Experimental realization of fermion superfluids in cold atoms is a major recent development in condensed matter physics¹². Manipulation of the parameters of Feshbach resonances and various trapping techniques provide control over the form and the strength of interaction and effective masses (band structure). In a wider context, having a mixture of bosonic and fermionic atoms and tuning to different parameter regimes, cold atom systems open possibilities to explore new exotic phases¹³.

Most experiments to date have exploited s-wave Feshbach resonances, but p-wave resonance, accessing a state nearly bound by a dipolar interaction, is also experimentally realizable¹⁴.

At strong enough coupling s-wave breached pairing is realized near Feshbach resonance, opening one or two spherical Fermi surfaces^{15,16}, while it is destroyed by instabilities in Meissner mass and number susceptibility¹⁶ at weak coupling. In this case, the p-wave breached superconductor is a stable ground state as we show in this work. Even when the s-wave interaction is repulsive, p-wave pairing instability may arise due to the Kohn-Luttinger effect¹⁷, in the same way it may support p-wave pairing in liquid ³He. In QCD with one flavor, the order parameter has total angular momentum one, i.e. $l = 1$ or $s = 1$, and the preferred phase exhibits color-spin-locking in the relativistic limit¹⁸.

Here we suggest analyze a model where the breached pair state arises in the p-wave channel. We find that in this context it is quite robustly stable. It seems not unreasonable, intuitively, that expanding an existing (lower-dimensional) locus of zeros into a two-dimensional surface should be significantly easier than producing, as in s-wave, a whole sphere of gapless excitations “from scratch”. In our context, we will show that pairing of mismatched Fermi surfaces can expand the locus of gapless states from a line to a torus (polar phase) or from two points into two lenticles (planar phase). We shall show that this phenomenon even occurs for arbitrarily small coupling and small Fermi surface mismatch, where our mean field approximation should be adequate. We presented a short account of some of this work previously¹⁹.

Anisotropic superfluid states that coexist with gapless modes at isolated points or lines in momentum space are also well known²⁰. Gapless states also are known to occur in the presence of magnetic impurities²¹ and, theoretically, in states with spontaneous breaking of translation symmetry²², where the gapless states span a two-dimensional Fermi surface. A similar phenomenon was found in two-dimensional d-wave superconductors subject to an external magnetic field²³. Strong coupling between different bands also may lead to zeros in quasiparticle excitations and gapless states²⁴.

A crucial difference between the model we consider and the conventional p-wave superfluid system, ³He, lies in the dissimilarity of the paired species. Although there are two components, there is no approximate quasispin symmetry,

and no analogue of the fully gapped B phase²⁵ arises.

II. ANISOTROPIC P-WAVE PAIRING

We consider a model system with the two species of fermions having the same Fermi velocity v_F , but different Fermi momenta $p_F \pm I/v_F$. The effective Hamiltonian is

$$H = \sum_{\mathbf{p}} [\epsilon_p^A a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \epsilon_p^B b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} - \Delta_{\mathbf{p}}^* a_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger + \Delta_{\mathbf{p}} b_{-\mathbf{p}} a_{\mathbf{p}}] \quad (1)$$

with $\epsilon_p^A = \xi_p + I$, $\epsilon_p^B = \xi_p - I$, $\xi_p = v(p - p_F)$, $\Delta_{\mathbf{p}} = \sum_{\mathbf{k}} V_{\mathbf{p}-\mathbf{k}} \langle a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger \rangle$. Here the attractive interspecies interaction is $-V_{\mathbf{p}-\mathbf{k}}$ within the “Debye” energy $2\omega_D$ around the Fermi surface ($\omega_D \gg I$), and vanishes at larger energies. The intraspecies interaction is assumed to be either repulsive or negligibly small. For the sake of simplicity, we have taken the gap function $\Delta_{\mathbf{p}}$ to be real. Excitations of the Hamiltonian (1) are gapless, $E_{\mathbf{p}} = \pm \sqrt{\xi_p^2 + \Delta_{\mathbf{p}}^2} + I$, provided that there are areas on the Fermi surface where $I > \Delta_{\mathbf{p}}$. The gap equation at zero temperature,

$$\Delta_{\mathbf{p}} = \frac{1}{2} \sum_{\mathbf{k}} V_{\mathbf{p}-\mathbf{k}} \frac{\Delta_{\mathbf{k}}}{\sqrt{\xi_k^2 + \Delta_{\mathbf{k}}^2}} \theta \left(\sqrt{\xi_k^2 + \Delta_{\mathbf{k}}^2} - I \right), \quad (2)$$

can be simplified by taking the integral over $d\xi_k$,

$$\Delta_{\mathbf{n}} = \nu \int \frac{d\Omega_{\mathbf{n}'}}{4\pi} V(\mathbf{n}, \mathbf{n}') \Delta_{\mathbf{n}'} \left(\ln \frac{2\omega_D}{|\Delta_{\mathbf{n}'}|} + \Theta(I - \Delta_{\mathbf{n}'}) \ln \frac{|\Delta_{\mathbf{n}'}|}{I + \sqrt{I^2 - \Delta_{\mathbf{n}'}^2}} \right). \quad (3)$$

Due to hierarchy of scales, $\omega_D \ll E_F$, the integration is performed in the narrow window around the Fermi surface (BCS approximation), i.e. $d^3k \rightarrow \nu d\Omega d\xi_k$, where $\nu = 1/(2\pi)^3 \int d^3k \delta(\xi_k) = k_F^2/(2\pi^2 v)$ is the density of states. In deriving Eq. (3) we neglected dependence of $V_{\mathbf{p}-\mathbf{k}}$ on the absolute values of \mathbf{p} and \mathbf{k} , this is a good approximation as long as the “Debye” energy is small compared to the Fermi energy, $\omega_D \ll E_F$.

We concentrate on the first harmonic (p -wave pairing) in the expansion of the interaction potential over the spherical functions, $V(\mathbf{n}, \mathbf{n}') = g(\mathbf{n} \cdot \mathbf{n}')$, with $g > 0$. The gap equation (3) allows two solutions corresponding to the value of the projection of the angular momentum of the Cooper pair onto some axis \mathbf{z} : $m = 0$ (polar phase), and $m = \pm 1$ (planar phase). We will now analyze these two solutions in detail.

A. Polar phase: $\Delta_{\mathbf{n}} \sim Y_{10}(\mathbf{n})$.

Let us look for a solution in the form $\Delta_{\mathbf{n}} = \Delta(\mathbf{z} \cdot \mathbf{n})$ where the direction \mathbf{z} reflects a broken rotational symmetry. In spherical coordinates, with $\mathbf{z} \parallel \mathbf{n}$ and $\theta = (\mathbf{n}, \mathbf{n}')$ the gap equation becomes,

$$-\frac{2}{\nu g} = \int_0^{\pi/2} d\theta \sin \theta \cos^2 \theta \ln \left(\frac{\Delta \cos \theta}{2\omega_D} \right) + \int_{\theta^*}^{\pi/2} d\theta \sin \theta \cos^2 \theta \ln \left(\frac{I + \sqrt{I^2 - \Delta^2 \cos^2 \theta}}{\Delta \cos \theta} \right). \quad (4)$$

where $\theta^* = \arccos(I/\Delta)$, for $I < \Delta$, and $\theta^* = 0$ for $I > \Delta$.

The solution of Eq. (4) is different in two cases of small and large Fermi momentum mismatches,

1. Small mismatch, $I < \Delta$.

Performing the angle integration in Eq. (4), we obtain

$$-\frac{1}{\nu g} = \frac{1}{3} \ln \left(\frac{\Delta}{2\omega_D} \right) - \frac{1}{9} + \frac{1}{6} \left(\frac{I}{\Delta} \right)^3 \frac{\pi}{2}. \quad (5)$$

At zero Fermi momentum mismatch, $I = 0$, the gap is, $\Delta_0/2\omega_D = \exp \left(\frac{1}{3} - \frac{3}{\nu g} \right) \approx 1.40 \exp \left(-\frac{3}{\nu g} \right)$. Using new dimensionless variables $x = I/\Delta_0$ and $y = \Delta/\Delta_0$ in Eq. (5) we obtain,

$$\ln \frac{1}{y} = \frac{\pi}{4} \left(\frac{x}{y} \right)^3. \quad (6)$$

At small $x \ll 1$, the solution of this equation is approximated by $y \approx 1 - \pi x^3/4$.

2. Large mismatch, $I > \Delta$.

The gap equation now takes the form,

$$-\frac{2}{\nu g} = \int_0^{\pi/2} d\theta \sin \theta \cos^2 \theta \ln \left(\frac{I + \sqrt{I^2 - \Delta^2 \cos^2 \theta}}{2\omega_D} \right). \quad (7)$$

Performing the angle integration we obtain

$$-\frac{1}{\nu g} = \frac{1}{3} \ln \left(\frac{I + \sqrt{I^2 - \Delta^2}}{2\omega_D} \right) - \frac{1}{9} - \frac{1}{6} \left(\frac{I}{\Delta} \right)^3 \sqrt{1 - \left(\frac{\Delta}{I} \right)^2} + \frac{1}{6} \left(\frac{I}{\Delta} \right)^3 \arcsin \left(\frac{\Delta}{I} \right), \quad (8)$$

which in the dimensionless variables $x = I/\Delta_0$, $y = \Delta/\Delta_0$ is simplified to,

$$\ln \frac{1}{x + \sqrt{x^2 - y^2}} = -\frac{1}{2} \left(\frac{x}{y} \right)^2 \sqrt{1 - \left(\frac{y}{x} \right)^2} + \frac{1}{2} \left(\frac{x}{y} \right)^3 \arcsin \left(\frac{y}{x} \right). \quad (9)$$

At $x = x_c = (1/2)e^{-1/3}$ the gap vanishes according to, $y \approx 1.55(x - x_c)^{1/2}$. The solution of the gap equation (6) and (9), consists of two branches shown in Fig. 1.

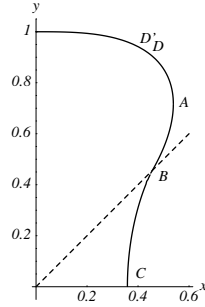


FIG. 1: The solutions $y(x)$ of Eqs. (6,9). The lower branch corresponds to the unstable state. Both branches merge at $\Delta = I$ (the point B; broken line). Non-zero solutions of the gap equation cease to exist beyond the point A. Stability conditions from the energy and the linear response give the points D and D', respectively. Coordinates of the characteristic points (for $\Delta_0 = 1$) are $A((4/3\pi e)^{1/3} = 0.538, e^{-1/3} = 0.717)$, $B(e^{-\pi/4} = 0.456, 0.456)$, $C(0, e^{-1/3}/2 = 0.358)$, $D(0.475, 0.886)$, $D'(0.440, 0.917)$.

B. Planar phase: $\Delta_{\mathbf{n}} \sim Y_{11}(\mathbf{n})$, $\Delta_{\mathbf{n}} \sim Y_{1-1}(\mathbf{n})$.

We now look for a solution corresponding to the momentum $m = \pm 1$ of a Cooper pair: $\Delta_{\mathbf{n}} = \Delta \sin(\mathbf{n}, \mathbf{z})e^{\pm i\phi}$, here ϕ is the polar angle in the plane perpendicular to \mathbf{z} . The gap equation becomes,

$$-\frac{2}{\nu g} = \int_0^{\pi/2} d\theta \sin^3 \theta \ln \left(\frac{\Delta \sin \theta}{2\omega_D} \right) + \int_0^{\theta^*} d\theta \sin^3 \theta \ln \left(\frac{I + \sqrt{I^2 - \Delta^2 \sin^2 \theta}}{\Delta \sin \theta} \right). \quad (10)$$

where $\theta^* = \arcsin(I/\Delta)$, for $I < \Delta$, and $\theta^* = \pi/2$, for $I > \Delta$. Again, we consider separately the regimes of small and large Fermi momentum mismatches.

1. *Small mismatch, $I < \Delta$.*

The first integral in Eq. (10) is simple: $\int_0^{\pi/2} d\theta \sin^3 \theta \ln [\sin \theta] = \frac{2}{3} \ln 2 - \frac{5}{9}$. The second integral in Eq. (10), denoted by $K_1(I)$, can be most easily calculated in the following way. Let us first differentiate $K_1(I)$ with respect to I ,

$$\frac{\partial K_1}{\partial I} = \int_0^{\arcsin(I/\Delta)} d\theta \frac{\sin^3 \theta}{\sqrt{I^2 - \Delta^2 \sin^2 \theta}} = -\frac{I}{2\Delta^2} + \frac{1}{4\Delta} \left(1 + \frac{I^2}{\Delta^2}\right) \ln \left(\frac{\Delta + I}{\Delta - I}\right). \quad (11)$$

Integrating now this equation over I we arrive at,

$$K_1(I) = \int_0^I dI \frac{\partial K_1}{\partial I} = -\frac{I^2}{6\Delta^2} + \frac{I}{4\Delta} \left(1 + \frac{I^2}{3\Delta^2}\right) \ln \left(\frac{\Delta + I}{\Delta - I}\right) + \frac{1}{3} \ln \left(1 - \frac{I^2}{\Delta^2}\right). \quad (12)$$

Substituting this expression into Eq. (10) we obtain the algebraic gap equation,

$$\ln [1/y] = -\frac{z^2}{4} + \frac{z}{8}(3 + z^2) \ln \left(\frac{1+z}{1-z}\right) + \frac{1}{2} \ln (1 - z^2), \quad (13)$$

where we utilized the same notations as before, $y = \Delta/\Delta_0$, $z = x/y = I/\Delta$, $x = I/\Delta_0$. The gap Δ_0 at zero mismatch is $\Delta_0/2\omega_D = \frac{1}{2} \exp(5/6 - 3/\nu g) \approx 1.15 \exp(-3/\nu g)$. For $x \ll 1$ the solution to the gap equation has the form, $y = 1 - 3x^4/4$. Note that the planar phase is more robust than the polar phase with respect to surviving the mismatch, in that the gap decreases as the fourth power for a planar phase instead of the third power for the polar phase.

Qualitatively, the behavior of the planar phase is very similar to Fig. 2 with the following numerical values of the characteristic points: $x_A = 0.674$, $y_A = 0.787$, $z_A = x_A/y_A = 0.856$; $x_C = e^{-5/6} = 0.435$.

2. *Large mismatch, $I > \Delta$.*

The gap equation at large mismatch becomes,

$$-\frac{2}{\nu g} = \int_0^{\pi/2} d\theta \sin^3 \theta \ln \left(\frac{I + \sqrt{I^2 - \Delta^2 \sin^2 \theta}}{2\omega_D}\right) \equiv K_2(I). \quad (14)$$

The integral here is calculated by the method used above [cf. Eq. (11)],

$$\frac{\partial K_2}{\partial I} = \int_0^{\pi/2} d\theta \frac{\sin^3 \theta}{\sqrt{I^2 - \Delta^2 \sin^2 \theta}} = -\frac{I}{2\Delta^2} + \frac{1}{4\Delta} \left(1 + \frac{I^2}{\Delta^2}\right) \ln \left(\frac{\Delta + I}{I - \Delta}\right). \quad (15)$$

Integrating this expression over I , and noting that $K_2(\Delta) = \frac{2}{3} \ln(\Delta/2\omega_D) + \frac{4}{3} \ln 2 - \frac{13}{18}$, we obtain,

$$K_2(I) = K_2(\Delta) + \int_{\Delta}^I dI \frac{\partial K_2}{\partial I} = \frac{2}{3} \ln \left(\frac{2\Delta}{2\omega_D}\right) - \frac{5}{9} - \frac{I^2}{6\Delta^2} + \frac{I}{4\Delta} \left(1 + \frac{I^2}{3\Delta^2}\right) \ln \left(\frac{\Delta + I}{I - \Delta}\right) + \frac{1}{3} \ln \left(\frac{I^2}{\Delta^2} - 1\right). \quad (16)$$

We observe that the gap equation in the form

$$\ln [1/y] = -\frac{z^2}{4} + \frac{z}{8}(3 + z^2) \ln \left|\frac{1+z}{1-z}\right| + \frac{1}{2} \ln |1 - z^2|, \quad (17)$$

is actually valid for any relation between I and Δ . We depict the solutions of the gap equation in the polar and planar phases in Fig. 2.

III. STABILITY OF THE OBTAINED SOLUTIONS

To analyze the stability of the obtained solutions one has to calculate the condensation energy for the two phases. The condensation energy is defined as the difference of the thermodynamic potentials in the superfluid Ω_s and normal Ω_n states²¹.

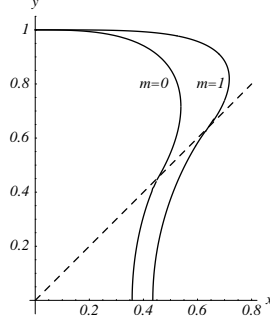


FIG. 2: Solutions $y(x)$ of the gap equation in the polar, $m = 0$, and planar, $m = 1$, phases. The lower branch corresponds to the unstable state. Non-zero solutions of the gap equation cease to exist beyond the point $y'(x) \rightarrow \infty$, where both branches merge. The dashed line corresponds to $z = 1$.

A. Condensation energy

The expectation value of the thermodynamic potential at zero temperature is

$$\begin{aligned}
\Omega_s &= \sum_{\mathbf{p}} \xi_p (u_p^2(n_{p,+} + n_{p,-}) + v_p^2(2 - n_{p,+} - n_{p,-})) + \sum_{\mathbf{p}} I_{\Delta}(n_{p,+} - n_{p,-}) \\
&\quad - \sum_{\mathbf{p}\mathbf{k}} V_{\mathbf{p}-\mathbf{k}} u_p v_p (1 - n_{p,+} - n_{p,-}) u_k v_k (1 - n_{k,+} - n_{k,-}) \\
&= \sum_{\mathbf{p}} \left(\xi_p - \sqrt{\xi_p^2 + \Delta_{\mathbf{p}}^2} (1 - n_{p,+} - n_{p,-}) \right) \\
&\quad + \sum_{\mathbf{p}} \frac{1}{2} \frac{\Delta_{\mathbf{p}}^2}{\sqrt{\xi_p^2 + \Delta_{\mathbf{p}}^2}} (1 - n_{p,+} - n_{p,-}) + \sum_{\mathbf{p}} I_{\Delta}(n_{p,+} - n_{p,-})
\end{aligned} \tag{18}$$

where we distinguish the mismatches I_{Δ}, I respectively with and without pairing. Here the u_p and v_p are parameters of the Bogoliubov transformation, $(u_p^2, v_p^2) = \frac{1}{2} \left(1 \pm \xi_p / \sqrt{\xi_p^2 + \Delta_{\mathbf{p}}^2} \right)$, and $n_{p,\pm}$ are the Fermi distributions with $E_p^{\pm} = \sqrt{\xi_p^2 + \Delta_{\mathbf{p}}^2} \pm I_{\Delta}$. The difference in the thermodynamic potentials in the superfluid and normal states is given by,

$$\begin{aligned}
\Omega_s - \Omega_n &= \sum_{|\xi_p| > I} |\xi_p| - \sum_{|\xi_p| > \sqrt{I_{\Delta}^2 - \Delta_{\mathbf{p}}^2}} \sqrt{\xi_p^2 + \Delta_{\mathbf{p}}^2} \\
&\quad + \sum_{|\xi_p| < I} I - \sum_{|\xi_p| < \sqrt{I_{\Delta}^2 - \Delta_{\mathbf{p}}^2}} I_{\Delta} + \sum_{|\xi_p| > \sqrt{I_{\Delta}^2 - \Delta_{\mathbf{p}}^2}} \frac{1}{2} \frac{\Delta_{\mathbf{p}}^2}{\sqrt{\xi_p^2 + \Delta_{\mathbf{p}}^2}}
\end{aligned} \tag{19}$$

which is readily simplified to

$$\Omega_s - \Omega_n = \nu \int \frac{d\mathbf{o}_{\mathbf{n}}}{4\pi} \left(-\frac{|\Delta_{\mathbf{n}}|^2}{2} - I_{\Delta} \sqrt{I_{\Delta}^2 - (\Delta_{\mathbf{n}})^2} \Theta(I_{\Delta} - |\Delta_{\mathbf{n}}|) + I^2 \right). \tag{20}$$

Note that $I = I_{\Delta}$ when pairing occurs at fixed chemical potentials, but when pairing is considered at fixed numbers of particles the chemical potentials, in general, are different before and after pairing: $I \neq I_{\Delta}$.

B. Polar phase

Applying the formula (20) to the polar phase at small mismatch, $z = I/\Delta < 1$, and fixed chemical potentials (so $I_{\Delta} = I$), we obtain

$$\Omega_s - \Omega_n = \nu \Delta^2 \left(-\frac{1}{6} - \frac{\pi z^3}{4} + z^2 \right), \tag{21}$$

which is negative provided that $z \leq 0.537$. This requires $y \geq 0.886$. Thus the upper branch $y(x)$ is stable for $x = I/\Delta_0 \leq 0.475$. This is the analog of Clogston-Chandrasekhar limit, here realized through an anisotropic p-wave interaction with the condensate. It is easy to verify that at large mismatches $z = I/\Delta > 1$, the condensation energy

$$\Omega_s - \Omega_n = \nu \Delta^2 \left(-\frac{1}{6} - \frac{z^3}{2} \arcsin(z^{-1}) - \frac{z}{2} \sqrt{z^2 - 1} + z^2 \right) > 0 \quad (22)$$

is always positive, meaning that the superfluid state is always unstable. We plot in Fig. 3 the normalized pressure difference between the superfluid state with the mismatch I and the normal state with $I = 0$ related to the difference at zero mismatch.

$$P(I) = -\frac{\Omega_s(I) - \Omega_n(0)}{\Omega_s(0) - \Omega_n(0)}.$$

It illustrates that the upper branch is stable for the mismatches less than the critical mismatch, where the dashed line crosses the solid line at $x = 0.75$. The upper branch is unstable for $x > 0.75$. The lower branch is always unstable.

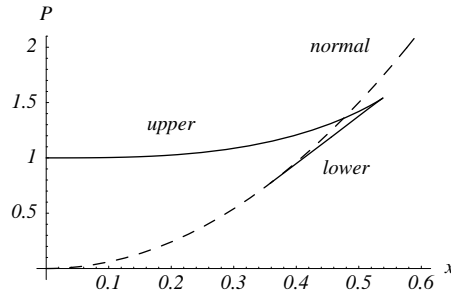


FIG. 3: Pressure as a function of the Fermi momenta mismatch $x = I/\Delta_0$ in the polar (solid line) and normal (broken line) phases. The upper and lower branches merge at a cusp point (point A at Fig.1). The lower branch corresponds to the unstable solution of the gap equation, it is tangent to the normal state at the point where the superfluid gap disappears $\Delta = 0$. When the pressure of the normal state exceeds the pressure of the polar phase (upper branch), there is a first order phase transition at $\Delta \neq 0$ from the superfluid to the normal state.

C. Planar phase

The condensation energy for the planar phase is easily evaluated with the help of the integral,

$$\int_0^{\arcsin a} d\theta \sin \theta \sqrt{a^2 - \sin^2 \theta} = \frac{a}{2} + \frac{a^2 - 1}{4} \ln \left(\frac{1+a}{1-a} \right), \quad 0 < a < 1, \quad (23)$$

and yields

$$\Omega_s - \Omega_n = \nu \Delta^2 \left(-\frac{1}{3} + \frac{z^2}{2} + \frac{z(1-z^2)}{4} \ln \left| \frac{1+z}{1-z} \right| \right). \quad (24)$$

For small mismatches, $z < 1$ the energy difference is positive for $z > 0.623$. For large mismatch $z > 1$, the condensation energy is always positive, $\Omega_s - \Omega_n > 0$, indicating that the lower branch is always unstable. The dependence of the normalized pressure for the planar phase is similar to Fig. 3.

For $I = 0$, we obtain the ratio of condensate in the planar phase to the one in polar phase to be $e/2 \approx 1.36$, indicating that the planar phase is the ground state at zero mismatch. This is in agreement with earlier analyses of $l = 1$ pairing^{26,27}; the latter contains an extension to higher harmonics. For our specific model Hamiltonian, at weak coupling, the planar phase is more stable.

IV. SUPERFLUID DENSITY, OR "MEISSNER" MASS

In the s-wave version of our model instability associated with spontaneous breakdown of translation symmetry^{7,8} can arise. It shows up as a negative superfluid density, or more formally as a negative coefficient multiplying the

gradient² terms in the effective Lagrangian for the superfluid mode. In the context of superconductivity, the gauge invariant form of this term encodes the Meissner mass² of the photon. So we can analyze this potential instability by checking whether the Meissner mass² is positive.

Following the standard methods in the theory of superconductivity²⁸ we calculate the super-currents in our system under the influence of a homogeneous in space gauge field \mathbf{A} , which is assumed to be transverse. The super-current is anisotropic, $j_i = \frac{e^2 N}{m} Q_{ik} A_k$ with the components obeying $Q_{xx} = Q_{yy}$. We consider two cases, when the field \mathbf{A} is perpendicular and parallel to the direction \mathbf{z} . It is straightforward to show that the superconducting gap is unaffected by the external field in the linear response regime.

Let us first consider the case $\mathbf{A} \cdot \mathbf{z} = 0$. The kernel is given by (for $T = 0$),

$$Q_{xx} = \lim_{q \rightarrow 0} \frac{2e^2 p_F^2 \nu}{m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\xi d\omega}{2\pi} \frac{d\mathbf{o}_n}{4\pi} \sin^2 \theta \cos^2 \phi \times \left(\frac{(\omega + \xi_{p+} - I)(\omega + \xi_{p-} - I) + \Delta_{p+} \Delta_{p-}}{((\omega + iI)^2 + \xi_{p+}^2 + \Delta_{p+}^2)((\omega + iI)^2 + \xi_{p-}^2 + \Delta_{p-}^2)} - \frac{1}{(i\omega - I - \xi_{p+})(i\omega - I - \xi_{p-})} \right), \quad (25)$$

where θ is the angle between \mathbf{p} and \mathbf{z} , and ϕ is the angle between \mathbf{p} and the plane containing the vectors \mathbf{z} and \mathbf{A} . The density of states $\nu = mp_F/2\pi^2$ can be taken independent of I (up to $O(I^2/E_F^2)$ corrections). Since the expression (25) is even in I , the sign of I is irrelevant, so in what follows we take $I > 0$ for simplicity. The second term in the parentheses represents the diamagnetic term (this is easy to verify by calculating first the integral over the frequency). Note, that each term in the parentheses is divergent at large ξ and ω , but their difference is convergent²⁸. Thus, the order of integrals cannot be changed for each term separately, but it can be changed when calculating the difference of both terms simultaneously²⁸.

It is convenient to take first the integral over $d\xi$ and then the integral over $d\omega$, since the second term in the parentheses is identically zero when integrating over $d\xi$. In the remaining term one can immediately assume $q = 0$ (the integral over $d\xi$ commutes with the transition $q \rightarrow 0$):

$$Q_{xx} = \frac{e^2 p_F^2 \nu}{2m^2} \int_0^\pi d\theta \sin^3 \theta \mathcal{J}_n, \quad \mathcal{J}_n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} d\xi \frac{(\omega + \xi - I)^2 + \Delta_n^2}{[\xi^2 + \Delta_n^2 + (\omega + iI)^2]^2}. \quad (26)$$

We will now make use of the following identity (which can be verified by integrating in parts),

$$\int_{-\infty}^{\infty} d\xi \frac{(\omega + \xi - I)^2 + \Delta_n^2}{[\xi^2 + \Delta_n^2 + (\omega + iI)^2]^2} = \frac{\Delta_n^2}{\Delta_n^2 + (\omega + iI)^2} \int_{-\infty}^{\infty} d\xi \frac{1}{\xi^2 + \Delta_n^2 + (\omega + iI)^2}. \quad (27)$$

The transformation (27) improves the convergence at large values of ξ and ω . One is now allowed to interchange the order of the integrals and calculate first the integral over the frequency with the help of,

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{[\Delta_n^2 + (\omega + iI)^2][\xi^2 + \Delta_n^2 + (\omega + iI)^2]} = \frac{1}{2\xi^2} \left(\frac{1}{|\Delta_n|} - \frac{1}{\sqrt{\xi^2 + \Delta_n^2}} \right) \Theta(|\Delta_n| - I) - \frac{1}{2\xi^2 \sqrt{\xi^2 + \Delta_n^2}} \Theta(I - |\Delta_n|) \Theta(\sqrt{\xi^2 + \Delta_n^2} - I). \quad (28)$$

Integration over $d\xi$ is now simple and gives,

$$\begin{aligned} \mathcal{J}_n &= \Theta(|\Delta_n| - I) \int_0^\infty d\xi \frac{\Delta_n^2}{\xi^2} \left(\frac{1}{|\Delta_n|} - \frac{1}{\sqrt{\xi^2 + \Delta_n^2}} \right) - \Theta(I - |\Delta_n|) \int_{\sqrt{I^2 - \Delta_n^2}}^\infty d\xi \frac{\Delta_n^2}{\xi^2 \sqrt{\xi^2 + \Delta_n^2}} \\ &= 1 - \frac{I}{\sqrt{I^2 - \Delta_n^2}} \Theta(I - |\Delta_n|). \end{aligned} \quad (29)$$

Substituting this expression into Eq. (26) we obtain,

$$Q_{xx} = \frac{3e^2 N}{4m} \int_0^\pi d\theta \sin^3 \theta \left(1 - \frac{I}{\sqrt{I^2 - \Delta_n^2}} \Theta(I - |\Delta_n|) \right) \quad (30)$$

In the case of $\mathbf{A} \parallel \mathbf{z}$, in complete analogy with the derivation of Eq. (30), we obtain,

$$Q_{zz} = \frac{3e^2 N}{2m} \int_0^\pi d\theta \sin \theta \cos^2 \theta \left(1 - \frac{I}{\sqrt{I^2 - \Delta_{\mathbf{n}}^2}} \Theta(I - |\Delta_{\mathbf{n}}|) \right), \quad (31)$$

where the integration over the angle ϕ gives 1, instead of 1/2 as for the Q_{xx} .

Combining both cases, $\mathbf{A} \perp \mathbf{z}$ and $\mathbf{A} \parallel \mathbf{z}$, we can now rewrite the general expression for Q_{ik} as follows,

$$\begin{pmatrix} Q_{zz} \\ Q_{xx} \end{pmatrix} = \frac{e^2 N}{m} \left(1 - \frac{3}{2} \int \frac{d\mathbf{o}_{\mathbf{n}}}{4\pi} \left(\frac{\cos^2 \theta}{\sin^2 \theta \cos^2 \phi} \right) \frac{I}{\sqrt{I^2 - \Delta_{\mathbf{n}}^2}} \Theta(I - |\Delta_{\mathbf{n}}|) \right). \quad (32)$$

We will now evaluate this expression for the two phases separately.

A. Polar phase

The integrals over the angles are easily calculated to yield, for small mismatches, $z = I/\Delta < 1$,

$$Q_{xx} = \frac{Ne^2}{m} \left[1 - \frac{3\pi z}{4} + \frac{3\pi z^3}{8} \right] \quad (33)$$

The homogeneous superconducting state is stable provided that this expression is positive $1 - 3\pi z/4 + 3\pi z^3/8 \geq 0$. This is so as long as $z \leq 0.480$, which implies that $y > 0.917$ or, equivalently, $x < 0.440$ (see Fig.4). For larger mismatches, $x > 0.440$, negative values of Q_{xx} probably indicate instability with respect to the transition into the Larkin-Ovchinnikov-Fulde-Ferrel state (with paired states of nonzero total momentum).

For large mismatches, $z = I/\Delta > 1$, the coefficient

$$Q_{xx} = \frac{Ne^2}{m} \left[1 - \frac{3z}{2} \arcsin(z^{-1}) + \frac{3z^3}{4} \arcsin(z^{-1}) - \frac{3z}{4} \sqrt{z^2 - 1} \right] \quad (34)$$

is always negative, indicating instability of the lower branch.

In the case $\mathbf{A} \parallel \mathbf{z}$, for small mismatches $z = I/\Delta < 1$, the effective density of superconducting fermions in the z -direction

$$Q_{zz} = \frac{Ne^2}{m} \left[1 - \frac{3\pi z^3}{4} \right] \quad (35)$$

is positive for $z \leq 0.752$, which implies $y > 0.717$ and $x < 0.589$. This is a weaker condition compared to the one obtained from the density in the x -direction Q_{xx} . For large mismatches $\Delta < I$, the coefficient

$$Q_{zz} = \frac{Ne^2}{m} \left[1 - \frac{3z^3}{2} \arcsin(z^{-1}) + \frac{3z}{2} \sqrt{z^2 - 1} \right] < 0, \quad (36)$$

again demonstrating that the lower branch is unstable.

B. Planar phase

For the planar phase the calculation of the integrals yields for both small and large mismatches,

$$\begin{pmatrix} Q_{zz} \\ Q_{xx} \end{pmatrix} = \frac{e^2 N}{m} \left[1 \mp \frac{3z^2}{4} - \frac{3z}{8} (1 \mp z^2) \ln \left| \frac{1+z}{1-z} \right| \right], \quad (37)$$

From these expressions we observe that Q_{xx} reaches zero at $z = 0.876$, while Q_{zz} always remains positive. Note that this critical value exceeds $z_A = 0.856$ and $z_D = 0.623$ meaning that in the energetically favorable state the density of superconducting fermions is always positive.

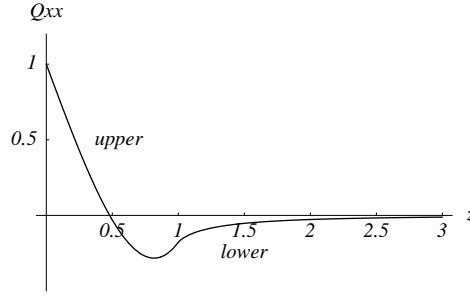


FIG. 4: Effective density (“Meissner mass”) of superconducting fermions in the x -direction, Q_{xx} , for the polar phase as a function of $z = I/\Delta$. The instability sets in at the point on the upper branch where Q_{xx} becomes negative. The lower branch is always unstable.

V. SPECIFIC HEAT

A definitive manifestation of the breached states with gapless excitations is the appearance of the term linear in temperature in the specific heat, which is characteristic for a normal Fermi liquid. The specific heat is given by the formula,

$$C = \sum_{\mathbf{p}} \left[E_{\mathbf{p}}^+ \frac{\partial n(E_{\mathbf{p}}^+)}{\partial T} + E_{\mathbf{p}}^- \frac{\partial n(E_{\mathbf{p}}^-)}{\partial T} \right], \quad (38)$$

where $E_{\mathbf{p}}^{\pm} = \sqrt{\xi_p^2 + \Delta_{\mathbf{n}}^2} \pm I$. At low temperatures $T \ll I$ the first term in Eq. (38) gives an exponentially small contribution which is negligible. The second term in Eq. (38) provides a leading contribution,

$$C = \frac{\nu}{4T^2} \int_{-\infty}^{\infty} d\xi \int \frac{d\phi_{\mathbf{n}}}{4\pi} \frac{\left(\sqrt{\xi^2 + |\Delta_{\mathbf{n}}|^2} - I \right)^2}{\cosh^2 \left[\frac{\sqrt{\xi^2 + |\Delta_{\mathbf{n}}|^2} - I}{2T} \right]}. \quad (39)$$

We will now evaluate this expression for both phases.

A. Polar phase

For the polar phase we have,

$$C = \frac{\nu}{4T^2} \int_{-\infty}^{\infty} d\xi \int_{-1}^1 \frac{dx}{2} \frac{\left(\sqrt{\xi^2 + \Delta^2 x^2} - I \right)^2}{\cosh^2 \left[\frac{\sqrt{\xi^2 + \Delta^2 x^2} - I}{2T} \right]}. \quad (40)$$

At low temperatures $T \ll \Delta$ the integral over the angle dx can be extended to infinity. Rescaling now $x\Delta = x'$ we observe that the integral is conveniently rewritten as the integral over $\rho = \sqrt{\xi_p^2 + x'^2}$,

$$C = \frac{\pi\nu}{4T^2\Delta} \int_0^{\infty} d\rho \rho \frac{(\rho - I)^2}{\cosh^2 \left[\frac{\rho - I}{2T} \right]}. \quad (41)$$

We note that the integrand in this expression is a very sharply peaked function at $\rho \approx I$ and can, therefore, write,

$$\int_0^{\infty} d\rho \rho \frac{(\rho - I)^2}{\cosh^2 \left[\frac{\rho - I}{2T} \right]} \rightarrow I \int_0^{\infty} d\rho \frac{(\rho - I)^2}{\cosh^2 \left[\frac{\rho - I}{2T} \right]} \rightarrow \frac{IT^3}{2} \int_{-\infty}^{\infty} \frac{dy y^2}{\cosh^2 [y/2]}. \quad (42)$$

The contribution of the gapless modes to the specific heat is thus,

$$C = \frac{\pi^2 \nu I}{6\Delta} T, \quad (43)$$

which is a fraction $I/4\Delta$ of the specific heat in the normal state.

B. Planar phase

The second term in Eq. (39) now yields,

$$C = \frac{\nu}{4T^2} \int_{-\infty}^{\infty} d\xi \int_0^{\pi/2} d\theta \sin \theta \frac{\left(\sqrt{\xi^2 + \Delta^2 \sin^2 \theta} - I\right)^2}{\cosh^2 \left[\frac{\sqrt{\xi^2 + \Delta^2 \sin^2 \theta} - I}{2T} \right]}. \quad (44)$$

This integral is a little more tricky for arbitrary ratio between I and Δ . However, since the principal contribution to the integral over $d\theta d\xi$ comes from the area where $\xi_p^2 + \Delta^2 \sin^2 \theta \approx I^2$, we conclude that for $I \ll \Delta$ the relevant angles are always small $\theta \ll 1$,

$$C = \frac{\nu}{4T^2} \int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\theta \theta \frac{\left(\sqrt{\xi^2 + \Delta^2 \theta^2} - I\right)^2}{\cosh^2 \left[\frac{\sqrt{\xi^2 + \Delta^2 \theta^2} - I}{2T} \right]}, \quad (45)$$

where again we can extend the limits of the angle integration to infinity due to fast convergence ($\Delta \gg T$) of the integral. Transforming this integral to the polar coordinates similar to the above ($\rho = \sqrt{\xi^2 + \Delta^2 \theta^2}$) one obtains,

$$C = \frac{\nu}{2T^2 \Delta} \int_0^{\infty} d\rho \rho^2 \frac{(\rho - I)^2}{\cosh^2 \left[\frac{\rho - I}{2T} \right]} \rightarrow \frac{\nu I^2}{2T^2 \Delta^2} \int_{-\infty}^{\infty} d\rho \frac{(\rho - I)^2}{\cosh^2 \left[\frac{\rho - I}{2T} \right]} = \frac{2\pi}{3} \frac{\nu I^2}{\Delta^2} T. \quad (46)$$

This result is valid provided that $T \ll I \ll \Delta$. As might be expected, the linear in T contribution to the specific heat is proportional to the area occupied by the gapless modes, i.e. the $\sim I/\Delta$ strip around the equator for the polar phase and the islands around the poles for the planar phase.

VI. COMPETITION OF THE ANISOTROPIC AND MIXED PHASES

In this section we analyze the stability of the obtained phases with respect to the transition into a spatially inhomogeneous state under the condition of *fixed number of particles* rather than at fixed chemical potentials as assumed in the preceding sections. Competition arises between a phase with the finite Fermi momentum mismatch I (i.e. with unequal numbers of particles) and a mixed phase consisting of spatially separated regions of superfluid state with zero mismatch (equal number of particles) and a normal (unpaired) phase accommodating the extra particles. To analyze the stability under the condition of a fixed number of particles one has to evaluate the energy (rather than the thermodynamic potential) of both phases.

A. Mixed phase

The energy of the mixed phase is given by the sum of energies of the superfluid and normal states, see Ref. (6) for the s wave,

$$E_{mix} = (1-x) \left(\frac{(6\pi^2 \bar{n})^{4/3} v}{4\pi^2} - \frac{(6\pi^2 \bar{n})^{2/3}}{2\pi^2 v} \frac{\Delta_0^2}{2c} \right) + x \left(\frac{(6\pi^2 \bar{n}_A)^{4/3} v}{8\pi^2} + \frac{(6\pi^2 \bar{n}_B)^{4/3} v}{8\pi^2} \right) \quad (47)$$

where $x/1-x$ fraction of the volume is in the normal/superfluid state (thermodynamic limit), total numbers of particles (densities) are $n_A = x\bar{n}_A + (1-x)\bar{n}$ and $n_B = x\bar{n}_B + (1-x)\bar{n}$, the superfluid state is given by the anisotropic

pairing at zero mismatch containing \bar{n} particles of each species. We introduced the constant c which assumes values $c = 3$ for the polar phase and $c = 3/2$ for the planar phase, respectively. There are two variational parameters, x and \bar{n} . It is convenient to introduce the new variables $\bar{n} = n_A + \delta\bar{n}$ and $\delta n = n_B - n_A$. At small mismatches we obtain up to the second order

$$E_{mix} = \frac{(6\pi^2 n_A)^{4/3} v}{4\pi^2} \left(1 + \frac{2}{3} \frac{\delta n}{n_A} + \frac{1}{9x} \frac{\delta n^2}{n_A^2} + \frac{2}{9} \frac{1-x}{x} \frac{\delta \bar{n}^2}{n_A^2} - \frac{2}{9} \frac{1-x}{x} \frac{\delta n \delta \bar{n}}{n_A^2} \right) - (1-x) \frac{(6\pi^2 n_A)^{2/3}}{2\pi^2 v} \frac{\Delta_0^2}{2c} \quad (48)$$

Subtracting the energy of the normal state,

$$E_n = \frac{(6\pi^2 n_A)^{4/3} v}{8\pi^2} + \frac{(6\pi^2 n_B)^{4/3} v}{8\pi^2} = \frac{(6\pi^2 n_A)^{4/3} v}{4\pi^2} \left(1 + \frac{2}{3} \frac{\delta n}{n_A} + \frac{1}{9} \frac{\delta n^2}{n_A^2} \right) \quad (49)$$

and minimizing $E_{mix} - E_n$ with respect to $\delta\bar{n}$ and x , we obtain

$$\begin{aligned} E_{mix} - E_n &= -\frac{(6\pi^2 n_A)^{2/3}}{2\pi^2 v} \frac{\Delta_0^2}{2c} (1 - x_{min})^2 \\ x_{min} &= \frac{(6\pi^2 n_A)^{1/3} v \sqrt{c} \delta n}{3\sqrt{2}\Delta_0 n_A} \end{aligned} \quad (50)$$

where Δ_0 is different for the polar, Δ_0^{pol} , and the planar, Δ_0^{pl} , phases.

B. Energy of the anisotropic phases

It is convenient to start from a thermodynamic potential at fixed chemical potentials and to perform a Legendre transformation to find the energy expressed in terms of fixed particle numbers. We begin by calculating the thermodynamic potential of the mismatched superfluid state,

$$\Omega_s(I_\Delta) - \Omega_0 = \sum_{\mathbf{p}} |\xi_{\mathbf{p}}| - \sum_{|\xi_{\mathbf{p}}| > \sqrt{I_\Delta^2 - \Delta_{\mathbf{p}}^2}} \sqrt{\xi_{\mathbf{p}}^2 + \Delta_{\mathbf{p}}^2} - \sum_{|\xi_{\mathbf{p}}| < \sqrt{I_\Delta^2 - \Delta_{\mathbf{p}}^2}} I_\Delta + \frac{1}{2} \sum_{|\xi_{\mathbf{p}}| > \sqrt{I_\Delta^2 - \Delta_{\mathbf{p}}^2}} \frac{\Delta_{\mathbf{p}}^2}{\sqrt{\xi_{\mathbf{p}}^2 + \Delta_{\mathbf{p}}^2}}, \quad (51)$$

counted from the potential of the normal state at zero mismatch, $\Omega_0 = \sum_{\mathbf{p}} \xi_{\mathbf{p}} - \sum_{\mathbf{p}} |\xi_{\mathbf{p}}|$; and we use the notation I_Δ for the mismatch in the superfluid state. The calculation similar to that from Section III gives,

$$\Omega_s(I_\Delta) = \Omega_0 + \nu \int \frac{d\mathbf{o}_{\mathbf{n}}}{4\pi} \left(-\frac{\Delta_{\mathbf{n}}^2}{2} - I_\Delta \sqrt{I_\Delta^2 - \Delta_{\mathbf{n}}^2} \Theta(I - |\Delta_{\mathbf{n}}|) \right). \quad (52)$$

We can now perform the Legendre transformation to find the energy of the superfluid state,

$$\begin{aligned} E_s &= \Omega_s + \frac{\mu_A + \mu_B}{2} (n_A + n_B) + \frac{\mu_A - \mu_B}{2} (n_A - n_B) \\ &= \Omega_0 + p_F (n_A + n_B) + \nu \int \frac{d\mathbf{o}_{\mathbf{n}}}{4\pi} \left(-\frac{\Delta_{\mathbf{n}}^2}{2} - I_\Delta \sqrt{I_\Delta^2 - \Delta_{\mathbf{n}}^2} \right) + I_\Delta \delta n, \end{aligned} \quad (53)$$

here $\delta n = n_B - n_A$ is the difference in the number of particles corresponding to the chemical potential mismatch I_Δ ,

$$\delta n = \sum_{|\xi_{\mathbf{p}}| < \sqrt{I_\Delta^2 - \Delta_{\mathbf{p}}^2}} \Theta(I_\Delta - \sqrt{\xi_{\mathbf{p}}^2 + \Delta_{\mathbf{p}}^2}) = 2\nu \int \frac{d\mathbf{o}_{\mathbf{n}}}{4\pi} \sqrt{I_\Delta^2 - \Delta_{\mathbf{n}}^2} \Theta(I - |\Delta_{\mathbf{n}}|). \quad (54)$$

The condensation energy (the difference in the energy of a superfluid state and a normal state) can now be written as,

$$E_s - E_n = \nu \int \frac{d\mathbf{o}_{\mathbf{n}}}{4\pi} \left(-\frac{\Delta_{\mathbf{n}}^2}{2} + I_\Delta \sqrt{I_\Delta^2 - \Delta_{\mathbf{n}}^2} - I^2 \right) \quad (55)$$

where we introduced the chemical potential mismatch $I = \delta n / 2\nu$ in the normal state, corresponding to the particle number mismatch δn . Note that in deriving Eq. (55) we also used that at non-zero mismatch, $E_n = E_0 + \nu I^2$ and $\Omega_n = \Omega_0 - \nu I^2$ in the normal state.

C. Polar phase

Applying the above expression (55) for the polar phase, $\Delta_{\mathbf{n}} = \Delta \cos \theta$,

$$E_s - E_0 = \nu \left(-\frac{\Delta^2}{6} + \frac{I_{\Delta}^3 \pi}{\Delta 4} \right), \quad (56)$$

where E_0 is the energy of the normal state with no mismatch, and the chemical potential mismatch I_{Δ} is related to the particle number difference δn as,

$$\delta n = \frac{\pi \nu I_{\Delta}^2}{2\Delta}. \quad (57)$$

We are interested in the case of small mismatches, $I_{\Delta} \ll \Delta_0$, where it is possible to approximate, $\Delta/\Delta_0 = 1 - \pi I_{\Delta}^4/4\Delta_0^4$. Using these expressions we obtain to the lowest non-vanishing order,

$$E_s - E_0 = -\nu \frac{\Delta_0^2}{6} \left[1 - 4\sqrt{\frac{2}{\pi}} \left(\frac{\delta n}{\nu \Delta_0} \right)^{3/2} \right] \quad (58)$$

Comparing it with the energy (50) of the mixed phase (which is simplified using, $\nu = (6\pi^2 n_A)^{2/3}/(2\pi^2 v)$ and $x_{min} = \sqrt{3/2} (\delta n/\nu \Delta_0)$),

$$E_{mix} - E_0 = -\nu \frac{\Delta_0^2}{6} \left[1 - \sqrt{6} \left(\frac{\delta n}{\nu \Delta_0} \right) \right] \quad (59)$$

we see that the homogeneous superfluid state is energetically favorable for small mismatches $\delta n \ll \nu \Delta_0$.

D. Planar phase

The energy of the planar phase, $\Delta_{\mathbf{n}} = \Delta \sin \theta e^{\pm i\phi}$, counted from the energy of the normal state with zero mismatch is

$$E_s - E_0 = \nu \left(-\frac{\Delta^2}{3} + \frac{I_{\Delta}^2}{2} - \frac{\Delta I_{\Delta}}{4} \left(1 - \frac{I_{\Delta}^2}{\Delta^2} \right) \ln \left[\frac{1 + I_{\Delta}/\Delta}{1 - I_{\Delta}/\Delta} \right] \right). \quad (60)$$

The particle difference is related to the chemical potential mismatch according to,

$$\delta n = \nu \left(I_{\Delta} - \frac{\Delta}{2} \left(1 - \frac{I_{\Delta}^2}{\Delta^2} \right) \ln \left[\frac{1 + I_{\Delta}/\Delta}{1 - I_{\Delta}/\Delta} \right] \right). \quad (61)$$

For small mismatches $\delta n \ll \nu \Delta_0$, one can approximate, $\delta n = 2\nu I_{\Delta}^3/3\Delta^2$. The gap at small mismatches has the form, $\Delta/\Delta_0 = 1 - 3I_{\Delta}^4/4\Delta^4$. Substituting this expression into Eq. (60) and eliminating δn with the help of Eq. (61) we obtain,

$$E_s - E_0 = -\nu \frac{\Delta_0^2}{3} \left[1 - 5 \frac{3^{4/3}}{2^{7/3}} \left(\frac{\delta n}{\nu \Delta_0} \right)^{4/3} \right] \quad (62)$$

Again, comparing this expression to the energy (50) of the mixed phase (with $x_{min} = \sqrt{3/2}(\delta n/\nu \Delta_0)$),

$$E_{mix} - E_0 = -\nu \frac{\Delta_0^2}{3} \left[1 - \sqrt{3} \left(\frac{\delta n}{\nu \Delta_0} \right) \right], \quad (63)$$

we conclude that the superfluid state is more stable than the mixed state.

VII. CONCLUSION

We have presented substantial theoretical evidence that our simple model supports planar phase gapless superfluidity in the ground state. For $I \ll \Delta$ the gapless modes contribute terms of high order in the mismatch, $\sim I^4$ in the solution of the gap equation and $\sim I^2$ in the heat capacity, i.e. they represent small perturbations. The planar phase is symmetric under simultaneous axial rotation and gauge (i.e., phase) transformation. The residual continuous symmetry of this state, and its favorable energy relative to plausible competitors (normal state, polar phase) suggest that it is a true ground state in this model. Also, we obtain a positive density of superfluid fermions, suggesting that inhomogeneous LOFF phases are disfavored at small I . Direct calculation of the energies in the anisotropic superfluid and mixed phases shows that the p-wave breached superfluidity is energetically favorable and the phase separation does not occur at small mismatch.

In some respects the same qualitative behavior we find here in the p wave resembles what arose in s wave¹⁰. Namely, isotropic s-wave superfluidity has two branches of solution: the upper BCS which is stable and – for simple interactions – fully gapped, and the lower branch which has gapless modes but is unstable. The striking difference is that in p wave the upper branch retains its stability while it develops two-dimensional lenticular surfaces of gapless modes. Specifically, the anisotropic p-wave breached pair phase, with coexisting superfluid and normal components, is stable already for a wide range of parameters at weak coupling using the simplest (momentum-independent) interaction. This bodes well for its future experimental realization.

In our model, which has no explicit spin degree of freedom, gapless modes occur for either choice of order parameter with residual continuous symmetry. By contrast, for ^3He in the B phase the p-wave spin-triplet order parameter is a 2×2 spin matrix, containing both polar and planar phases components, there are no zeros in the quasiparticle energies, and the phenomenology broadly resembles that of a conventional s-wave state²⁵; in the A phase (which arises only at $T \neq 0$ ²⁹) the separate up and down spin components pair with themselves, in an orbital p wave, and no possibility of a mismatch arises.

Application of anisotropic breached superfluidity to high density QCD may lead to a viable stable phase at low chemical potential. Though the gap with higher orbital harmonics is suppressed, the smallest gap defines neutrino emission properties and hence cooling rate of the neutron star³⁰.

It is possible that the emergent fermi gas of gapless excitations develops, as a result of residual interactions, secondary condensations. Also, one may consider analogous possibilities for particle-hole, as opposed to particle-particle, pairing. In that context, deviations from nesting play the role that fermi surface mismatch plays in the particle-particle case. We are actively investigating these issues.

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